

# Dynamical Mass in Strongly Coupled QED<sup>1</sup>

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## Abstract

We review dynamical mass generation in three and four dimensional Abelian gauge theories. The basic approach analyzed here is to solve Schwinger-Dyson equations in some approximations. In both cases, the coupling has to be larger than a critical value to have dynamical symmetry breaking. In  $QED_4$ , chiral symmetry is spontaneously broken if  $\alpha > \alpha_c (\simeq \pi/3)$  in ladder approximation. The electron becomes massive, and is realized as Skyrmion in the low energy effective action. In  $QED_3$ , electron gets dynamical mass in  $1/N$  expansion if  $N < N_c (\simeq 32/\pi^2)$ . With a Chern-Simons term, parity tends to break maximally in  $QED_3$ . The quantum phase structure of  $QED_3$  with or without a Chern-Simons term is peculiar.

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# 1 Introduction

The idea of spontaneous symmetry breaking constitutes the essential part of the Standard Model which is proven to be a right theory to describe Nature upto several  $100\text{GeV}$  energy scale. By the success of the idea, most of us believe that all different-looking forces in Nature can be explained by a single principle. One compelling way to break symmetry spontaneously is by dynamics. Namely, asymmetric ground state is dynamically favorable. We call this dynamical symmetry breaking (DSB) [1]. It seems the only way of spontaneous symmetry breaking that is known to be realized in Nature so far. The examples are the BCS theory of superconductivity, superfluid  $He^3$ , chiral symmetry breaking in QCD, *etc.*

It is obvious that we need strong coupling to achieve DSB.<sup>1</sup> For weak coupling, perturbation is good and vacuum in the perturbation theory respects the symmetry of Lagrangian. A naive explanation of DSB in strongly coupled system is following. Quantum fluctuation creates a virtual pair of fermion and antifermion. The pair may form a bound state virtually. But, if the binding energy of the bound state is larger than the energy needed to create such a pair at free state, the energy of the bound state will be lower than that of the perturbative vacuum. Then, it is energetically preferable to pump infinitely many bound states out of the perturbative vacuum, indicating the instability of the perturbative vacuum. The Bose condensate of the bound states defines a new vacuum, asymmetric if the condensate is not invariant under the symmetry of Lagrangian. The particle spectra will be quite different from the fields in the Lagrangian.

For several reasons, though there is no direct proof, we believe that DSB occurs in QCD; the chiral symmetry of QCD is spontaneously broken by strong interaction.

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<sup>1</sup>For BCS superconductivity, arbitrary weak attraction leads to instability of ground state due to the presence of Fermi surface.

The reasons are, among others, (1) there are 8 light pseudoscalar mesons (pions and kaons), compared to  $\Lambda_{\text{QCD}}$ , (2) lattice calculations[2] show  $\langle \bar{\Psi}\Psi \rangle \neq 0$ , and (3) the anomaly argument of 't Hooft-Coleman-Witten in  $1/N$  [3]. But, the full understanding of DSB in QCD is still lacking. Understanding how chiral symmetry is broken in QCD is important not only in hadron physics but also in the evolution of our Universe or in the proposed heavy ion collision experiment or in constructing models for dynamical breaking of electroweak symmetry. There are several models studied to understand DSB in a simplified setup but containing the general feature of QCD. They are Nambu-Jona-Lasinio model [4], strongly coupled quenched QED [5], three dimensional large  $N_f$  QED [6], three dimensional Thirring model [7], *etc.* All the models show that DSB occurs in a strong phase.

In this lecture, I will concentrate on strongly coupled QED in four and three dimensions. As we will see later, in the strong phase of QED, a fermion bilinear operator gets large anomalous mass dimension and the wave function of the electron-positron bound state collapses due to the strong coupling in deep UV region, leading to spontaneous breaking of chiral symmetry. This phenomena of strongly coupled QED is used in several models for dynamical breaking of the electroweak symmetry as in the top-quark condensate model and in the walking technicolor model [8].

## 2 Strongly coupled Quantum Electrodynamics

Since in QED the vacuum polarization screens the electric charge at long distance, the electric charge becomes stronger and stronger in deep UV region, which leads to so-called “Moscow zero” or “Landau ghost” [9]. It means that the renormalized electric charge has to be zero in order that QED may have well-defined limit as we remove the UV cut-off of the theory; QED is a trivial theory. If QED is an effective theory below a scale,  $\Lambda < \Lambda_{\text{QED}}$ , then we do not have to worry about the triviality

of QED. Another way to avoid this triviality problem in QED is that QED has a nontrivial UV fixed point. The latter possibility was studied by several people, starting from the program of Baker, Johnson and Willey and Adler, known as finite QED [10]. Near a fixed point, if it exists, the running effect of the coupling will be unimportant, thus ladder approximation is applicable. Maskawa and Nakajima [11] studied the Schwinger-Dyson (SD) equation for the fermion propagator in the ladder approximation in QED. They found chiral symmetry is spontaneously broken if the coupling is larger than a critical coupling,  $\alpha > \alpha_c$ . ( $\alpha_c = \frac{\pi}{3}$  in Landau gauge.) The explicit analysis for the SD-equation for the fermion propagator was done by Fukuda and Kugo [12]. Fomin, Gusynin and Miransky analyzed the Bethe-Salpeter (BS) equations for spinless bound state in quenched QED and showed that for a supercritical coupling the bound state wave function collapses, which then leads to chiral symmetry breaking [13].

Bardeen, Leung and Love [14] argued that in the strong phase of QED a four-Fermi operator becomes relevant due to the large anomalous dimension of  $\bar{\psi}\psi(x)$ . Soon after, with adding the chirally invariant operator

$$\frac{g^2}{\Lambda^2} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 \right], \quad (1)$$

it is found that there is a critical line in the coupling space of  $(g^2, \alpha)$  for chiral symmetry breaking [15]. Hong and Rajeev analyzed the solutions to the SD equations in quenched QED in the language of the operator product expansion and found that the solutions consistently describe spontaneous chiral symmetry breaking in the strong phase of QED [16]. They also constructed an effective action for the strongly coupled QED with spontaneous chiral symmetry breaking, and argued that the bosonized QED admits a stable fermionic soliton, which can be identified as a massive electron, and thus the strong QED does not exhibit confinement, even though chiral symmetry breaking does occur [17].

## 2.1 The Schwinger-Dyson equation in the quenched QED

We derive the gap equation for QED. One of the Schwinger-Dyson equations for QED provides the relation between the full fermion propagator, the photon propagator, and the three-point vertex, which is in euclidean notation

$$\not{p}A(p^2) - B(p^2) = \not{p} - m_0 + e \int_k \gamma^\mu D_{\mu\nu}(p-k) \frac{-\not{k}A - B}{A^2k^2 + B^2} \Lambda^\nu \quad (2)$$

We approximate the photon propagator by free propagator in Landau gauge, and the vertex by the tree level value,  $e\gamma^\nu$ . The resulting equation then may be separated into two equations according to the pieces with different spinor matrix structures.

$$B(p^2) = m_0 + 3e^2 \int_k \frac{1}{(p-k)^2} \frac{B}{A^2k^2 + B^2} \quad (3)$$

$$A(p^2) = 1 - \frac{e^2}{p^2} \int_k \frac{g_{\mu\nu} - (p-k)_\mu(p-k)_\nu/(p-k)^2}{(p-k)^2} \text{tr} \not{p} \gamma^\mu \not{k} \gamma^\nu \frac{A}{A^2k^2 + B^2} \quad (4)$$

Taking trace over the gamma matrices, the wave function renormalization constant becomes

$$A(p^2) = 1 + 4 \frac{e^2}{p^2} \int_k \left( \frac{p \cdot k}{(p-k)^2} + 2 \frac{p \cdot (p-k) k \cdot (p-k)}{(p-k)^4} \right) \frac{A}{A^2k^2 + B^2} \quad (5)$$

Note that the angular integration of eq. (5) vanishes

$$\int_0^\pi \sin^2 \theta d\theta \left\{ \frac{pk \cos \theta}{(p^2 + k^2 - 2pk \cos \theta)} + 2 \frac{(p^2 - pk \cos \theta)(pk \cos \theta - k^2)}{(p^2 + k^2 - 2pk \cos \theta)^2} \right\} = 0 \quad (6)$$

which would not be true if the gauge fixing parameter  $\xi \neq \infty$ . Therefore, for ladder approximation in Landau gauge,  $A(p^2) = 1$  and

$$B(p^2) = m_0 + 3e^2 \int_k \frac{1}{(p-k)^2} \frac{B}{k^2 + B^2} \quad (7)$$

Using the formular

$$\int_0^\pi \sin^2 \theta d\theta \frac{1}{(p^2 + k^2 - 2pk \cos \theta)} = \frac{\pi}{2pk} \left[ \frac{k}{p} \theta(p-k) + \frac{p}{k} \theta(k-p) \right] \quad (8)$$

eq. (7) becomes, after angular integration,

$$B(p^2) = m_0 + \frac{3\alpha}{4\pi} \int dk^2 \frac{k^2 B}{k^2 + B^2} \left[ \frac{\theta(p^2 - k^2)}{p^2} + \frac{\theta(k^2 - p^2)}{k^2} \right] \quad (9)$$

One may convert the above integral equation into a differential equation. We differentiate eq.(9) with respect to  $p^2$  to obtain

$$\frac{dB}{dp^2} = -3 \frac{\alpha}{4\pi(p^2)^2} \int_0^{p^2} dk^2 \frac{k^2 B}{k^2 + B^2} \quad (10)$$

Multiplying  $(p^2)^2$  and differentiating once again we obtain

$$\frac{d}{d(p^2)} \left( (p^2)^2 \frac{dB}{dp^2} \right) = \frac{-3\alpha}{4\pi} \frac{p^2 B}{p^2 + B^2}. \quad (11)$$

Converting the integral equation into a differential equation, we get two boundary conditions:

$$\lim_{p^2 \rightarrow 0} \left( (p^2)^2 \frac{dB(p^2)}{dp^2} \right) = 0 \quad (12)$$

$$\lim_{p^2 \rightarrow \Lambda^2} \left[ p^2 \frac{dB(p^2)}{dp^2} + B(p^2) \right] = m_0, \quad (13)$$

where  $\Lambda$  is a UV cut-off.

Letting  $p^2 = x$  and  $\omega = \sqrt{1 - 3\alpha/\pi}$ , we find

$$x \frac{d^2 B}{dx^2} + 2 \frac{dB}{dx} + \frac{(1 - \omega^2)}{4} \frac{B}{x + B^2} = 0 \quad (14)$$

One may linearize the above equation as

$$x \frac{d^2 B}{dx^2} + 2 \frac{dB}{dx} + \frac{(1 - \omega^2)}{4} \frac{B}{x + m^2} = 0 \quad (15)$$

where  $m = B(0)$ . This is a good approximation in both infrared ( $p^2 \ll m^2$ ) and ultraviolet ( $p^2 \gg m^2$ ) regions. The numerical analysis shows that it is good even at  $p^2 \sim m^2$ .

Putting  $z = -x/m^2$  and  $Y = B(x)/m$ , we get

$$z(1 - z) \frac{d^2 Y}{dz^2} + (2 - 2z) \frac{dY}{dz} - \frac{1}{4} (1 - \omega^2) Y = 0. \quad (16)$$

The general solution to the above equation (16) is a linear combination of two independent hypergeometric functions

$$Y(z) = C_1 {}_2F_1(a, b, 2; z) \quad (17)$$

$$+ C_2 [(-z)^{-a} {}_2F_1(a, -b, 2b; z) + (-z)^{-b} {}_2F_1(b, -a, 2b; z)] , \quad (18)$$

where  $a = (1 + \omega)/2$ ,  $b = (1 - \omega)/2$  and the hypergeometric function is in a series form

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}. \quad (19)$$

Since, as  $p^2 \rightarrow 0$ ,  $B \rightarrow m$ ,  $Y(0) = 1$ . Therefore  $C_1 = 1$  and  $C_2 = 0$ , namely

$$B(p^2) = m {}_2F_1\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}, 2; -p^2/m^2\right) \quad (20)$$

The UV boundary condition yields a relation between  $m$ ,  $m_0$  and  $\Lambda$ ;

$$m_0 \simeq m \frac{\Gamma(\omega)}{\Gamma\left(\frac{\omega+1}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)} \left(\frac{\Lambda}{m}\right)^{\omega-1}; \quad \text{at } \alpha < \frac{\pi}{3}, \quad (21)$$

$$m_0 \simeq m \frac{4}{\pi} \left(\ln \frac{\Lambda}{m} + 2 \ln 2 - 1\right) \left(\frac{\Lambda}{m}\right)^{-1}; \quad \text{at } \alpha = \frac{\pi}{3} \quad (22)$$

$$m_0 \simeq \frac{m^2}{\Lambda} \left(\frac{2 \coth(\pi \bar{\omega})}{\pi \bar{\omega}}\right)^{1/2} \sin\left(\bar{\omega} \ln \frac{\Lambda}{m} + \theta(\bar{\omega})\right); \quad \text{at } \alpha > \frac{\pi}{3}, \quad (23)$$

where  $\bar{\omega} = \sqrt{3\alpha/\pi - 1}$  and  $\theta(\bar{\omega}) = \text{Arg}(\Gamma(1 + \omega)/\Gamma^2((1 + \omega)/2))$ . We see that for  $\alpha \leq \pi/3$  no dynamical mass is generated for a finite  $\Lambda$  in the chiral limit ( $m_0 \rightarrow 0$ ).

On the other hand, for  $\alpha > \pi/3$ , there is a dynamically generated mass if

$$\sin\left(\bar{\omega} \ln \frac{\Lambda}{m} + \theta(\bar{\omega})\right) = 0. \quad (24)$$

Near the critical coupling,  $\theta(\bar{\omega}) \simeq 0$  and the dynamically generated mass is

$$m \simeq \Lambda \exp(-\pi n/\bar{\omega}), \quad n = 1, 2, \dots \quad (25)$$

For  $\Lambda \rightarrow \infty$ , we want  $m$  to remain finite. This is possible only if the coupling has a nontrivial dependence in  $\Lambda$ : for  $n = 1$ <sup>2</sup>

$$\alpha(\Lambda) = \alpha_c + \frac{\alpha_c \pi^2}{\left[\ln \frac{\Lambda}{\mu}\right]^2}, \quad (26)$$

where  $\mu$  is some scale we introduced by hand as a dimensional transmutation (Note that if we remove the cut-off the theory has no scale.) The corresponding  $\beta$  function is then

$$\beta(\alpha) \equiv \Lambda \frac{\partial}{\partial \Lambda} \alpha = -\frac{2}{\pi \sqrt{\alpha_c}} (\alpha - \alpha_c)^{3/2} \quad (27)$$

We see that the  $\beta$  function is a nonanalytic function of the coupling constant as indication that the origin of this  $\beta$  function is nonperturbative. As Miransky pointed out this nonperturbative running coupling originates from the wave function collapse for the electron-positron bound state when  $\alpha > \alpha_c$  [18].

## 2.2 Vacuum energy

The Schwinger-Dyson equation is derived from the stationary condition for the effective action of a composite operator  $G(x, y; K)$  for vanishing source  $K$ :

$$\frac{\delta \Gamma}{\delta G(x, y)} = 0 \quad (28)$$

Therefore, to find the true vacuum solution, one has to check the stability condition.

Define an effective vacuum energy  $V(G)$

$$\Gamma(G) = -V(G) \int d^4x \quad (29)$$

where we use the translational invariance of the two-point function. Then, the effective vacuum energy is, for the quenched QED,

$$\begin{aligned} V(G) = & -\frac{1}{4\pi^2} \int p^2 dp^2 \left[ \frac{1}{2} \ln \left( 1 + \frac{B^2(p)}{p^2} \right) + \frac{p^2}{p^2 + B^2(p)} - 1 \right] \\ & - \frac{3e^2}{128\pi^4} \int dp^2 dk^2 \frac{p^2 B(p) k^2 B(k)}{[p^2 + B^2][k^2 + B^2]} \left\{ \frac{\theta(p^2 - k^2)}{p^2} + \frac{\theta(k^2 - p^2)}{k^2} \right\} \end{aligned} \quad (30)$$

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<sup>2</sup>In fact,  $n = 1$  corresponds to a correct bound-state wavefunction.



where we have normalized the vacuum energy such that  $V(B = 0) = 0$ . One can easily see that any nontrivial solution ( $B(p) \neq 0$ ) has a lower energy than the trivial solution,  $B(p) = 0$ . Therefore, chiral symmetry is spontaneously broken once chiral-symmetry breaking solution is found, which we have seen to exist for  $\alpha > \pi/3$  in quenched QED.

### 2.3 Nambu-Goldstone bosons and wave-function collapse

For  $N(\geq 2)$  identical “electrons”, the Lagrangian has  $SU(N)_L \times SU(N)_R \times U(1)_V$  symmetry. From the Ward-Takahashi identity for the axial current inserted in a two-point function,

$$\begin{aligned} \partial_z^\mu \langle 0 | T j_\mu^\alpha(z) \psi(x) \bar{\psi}(y) | 0 \rangle &= \frac{i}{2} \langle 0 | T \bar{\psi}(z) \{m_0, \lambda^\alpha\} \psi(z) \psi(x) \bar{\psi}(y) | 0 \rangle \\ &+ i\delta^4(x-z) \left\langle 0 \left| T \left( i\frac{\lambda}{2} \psi(x) \bar{\psi}(y) \right) \right| 0 \right\rangle \\ &+ i\delta^4(x-z) \left\langle 0 \left| T \left( \psi(x) \bar{\psi}(y) (-i\frac{\lambda}{2}) \right) \right| 0 \right\rangle \end{aligned} \quad (31)$$

where  $\lambda^\alpha/2$ 's are the generators for  $SU(N)$ , one can derive in the massless limit a relation in momentum space

$$P^\mu \Gamma_{5\mu}^\alpha(p, k; P) = -G^{-1}(p) \frac{\lambda^\alpha}{2} \gamma_5 - \frac{\lambda^\alpha}{2} \gamma_5 G^{-1}(k). \quad (32)$$

where the vertex in momentum space is defined as

$$iG(p) \Gamma_{5\mu}^\alpha iG(k) \delta^4(p-k-P) = \int dx dy dz e^{ipx-iky-iPz} \langle 0 | T j_{5\mu}^\alpha(z) \psi(x) \bar{\psi}(y) | 0 \rangle \quad (33)$$

Note near the pole

$$\begin{aligned} \tilde{\Gamma}_{5\mu}(x, y) &= \langle 0 | j_{5\mu}^\alpha(0) | P, \beta \rangle \frac{-i}{P^2 + i\epsilon} \langle P, \beta | \psi(x) \bar{\psi}(y) | 0 \rangle \\ &= iF P_\mu \delta^{\alpha\beta} \frac{-i}{P^2 + i\epsilon} \chi^{P;\beta}(x, y) \end{aligned} \quad (34)$$

where  $F$  is the “electropion” decay constant. When  $p \rightarrow k$  (or  $P \rightarrow 0$ )

$$G(k) P^\mu \Gamma_{5\mu}^\alpha(k, k; P) G(k) = F \chi^{P;\alpha}(k) \quad (35)$$

with

$$\chi(x, y) = \chi(x - y) = \int_k e^{ik \cdot (x-y)} \chi^{P;\beta}(k). \quad (36)$$

One finds in the chiral limit

$$G^{-1}(q) \chi^{P;\alpha} G^{-1}(q) = -\{G^{-1}(q), \frac{\lambda^\alpha}{2} \gamma_5\} \quad (37)$$

Or, in the case at hand where  $A(q) = 1$ ,

$$(q - B(q^2)) \chi^{P;\alpha}(q) (q - B(q^2)) = \lambda^\alpha B(q^2) \gamma_5. \quad (38)$$

We may expand the amplitude for the bound state in the Dirac matrices

$$\chi^{P;\alpha}(q) = [\chi_1^P + \chi_2^P \not{P} + \chi_3^P \not{q} + \chi_4^P \sigma^{\mu\nu} (P_\mu q_\nu - P_\nu q_\mu)] \frac{\lambda^\alpha}{2} \gamma_5. \quad (39)$$

As  $P \rightarrow 0$ , the pseudo-scalar channel survives to give

$$\chi_1^P(q) = \frac{2}{F} \frac{B(q^2)}{q^2 + B^2(q^2)}. \quad (40)$$

We see that the electron self-energy is closely related to the bound-state wave function.

Without going into the WT identity, one can also see the relation directly from the Bethe-Salpeter (BS) equation;

$$\left(\frac{1}{2} \not{P} + \not{q} - m_a\right)_{nn_1} \chi^P(q)_{n_1 m_1} \left(\frac{1}{2} \not{P} - \not{q} - m_a\right)_{m_1 m} = \int_q K_{mn, n_2 m_2}(q - k) \chi^P(k)_{n_2 m_2}. \quad (41)$$

The BS kernel is given as

$$K_{mn, n_2 m_2}(q - k) = e^2 (\gamma^\mu)_{nn_2} D_{\mu\nu}(q - k) (\gamma^\nu)_{m_2 m} + e^2 (\gamma^\mu)_{nm} (\gamma^\nu)_{m_2 n_2} D_{\mu\nu}(P), \quad (42)$$

where we have included the dynamically generated fermion mass,  $m_a$ .

For the pseudo-scalar channel with  $P \rightarrow 0$  one gets

$$(m^2 + q^2) \chi_{ab;1}^P(q) = 12\pi\alpha \int_k \frac{1}{(q - k)^2} \chi_{ab;1}^P(k). \quad (43)$$

where we take  $m_a = m_b = m$ . (More precisely, we should take  $B(q^2)$  for  $m_a$  and  $m_b$ , but taking  $m$  for fermion self energy is a good approximation in the region we are interested in.) Note that the second term in the kernel, corresponding to annihilation process, does not contribute to the BS equation in ladder approximation. If we let

$$(m^2 + q^2)\chi_{ab;1}^P(q) = \text{const.} \times B(q^2), \quad (44)$$

we see that  $B(q^2)$  satisfies the linearized gap equation

$$B(q^2) = 12\pi\alpha \int_k \frac{1}{(q-k)^2} \frac{B(k^2)}{k^2 + m^2} \quad (45)$$

This equation has been solved exactly. But, here we solve it again in the position space to find the physical meaning of the oscillating solutions for a supercritical coupling. The BS equation eq.(43) is after Fourier-transforming to the position space

$$(m^2 - \partial^2)\chi^P(x) = \frac{3\alpha}{\pi} \frac{1}{r^2} \chi(x) \quad (46)$$

where  $r = \sqrt{x_i x_i}$ . Eq.(46) can be thought of as a Schrödinger equation for a bound-state in four-dimensional euclidean space with an attractive potential

$$\left(-\partial^2 - \frac{3\alpha}{\pi} \frac{1}{r^2}\right) \chi^P(x) = -m^2 \chi^P(x) \quad (47)$$

Since large  $m$  ( $\simeq B(0)$ ) corresponds to the solution of lower vacuum energy to the gap equation, we look for the ground state of the Schrödinger equation, eq. (46). Because of the scale invariance, we see that if  $\chi^P(x)$  is a solution to the Schrödinger equation with eigen value  $-m^2$ , so is  $\chi^P(\lambda x)$  with eigen value  $-m^2/\lambda^2$ . Therefore the spectrum of the Schrödinger equation is continuous. If there is a bound state, the ground state energy has to be  $-\infty$ . In quantum mechanics, we know that most of time particle exists in the region where the energy of the particle is larger than

the potential energy. Thus, the ground state wave function collapses toward the origin. To make sense out of this divergence, we regularize the potential as follows:

$$V(r) = \begin{cases} -\frac{\alpha/\alpha_c}{r^2}, & \text{if } r \geq a, \\ -\frac{\alpha/\alpha_c}{a^2}, & \text{otherwise,} \end{cases} \quad (48)$$

where  $a$  is a short-distance cut-off and  $\alpha_c = \frac{\pi}{3}$ . At short distance,  $r \ll 1/m$ , one may neglect the  $m^2$  term. Then, the  $O(4)$ -invariant solution satisfies

$$\chi^{P''}(r) + \frac{3}{r}\chi^{P'}(r) - V(r)\chi^P(r) = 0. \quad (49)$$

When  $r \geq a$ ,

$$\chi^P(r) = Ar^{s_1} + Br^{s_2} \quad (50)$$

with  $s_1 = s_2^* = -1 + i\sqrt{\alpha/\alpha_c - 1}$ . Similarly, when  $0 \leq r \leq a$ ,

$$\chi^P(r) = \frac{C}{r}J_1(kr), \quad (51)$$

where  $C$  is a constant,  $k = \sqrt{\alpha/\alpha_c} a^{-1}$ , and  $J_1$  is the Bessel function of the first kind. The boundary condition that both of  $\chi^P(r)$  and its derivative be continuous at  $r = a$  leads to

$$\frac{B}{A} = e^{2i\theta} a^{s_1 - s_2} \quad (52)$$

with

$$\theta = \text{Arg} \left( \sqrt{\alpha/\alpha_c - 1} + i\sqrt{\alpha/\alpha_c} \frac{J_1'(\sqrt{\alpha/\alpha_c})}{J_1(\sqrt{\alpha/\alpha_c})} \right). \quad (53)$$

Then, for  $a \leq r \ll 1/m$ ,

$$\chi^P(r) = \frac{\tilde{A}}{r} \cos \left[ \sqrt{\alpha/\alpha_c - 1} \ln \left( \frac{r}{a} \right) + \theta \right], \quad (54)$$

where  $\tilde{A}$  is a constant. Now, if we let  $a$  go to zero, the wave function oscillates rapidly and the number of zeros increases exponentially, which implies the energy of

the bound state is  $-\infty$  as we argued earlier. To regularize this divergence, we have to let the coupling constant run as we change the cut-off such that

$$\lim_{a \rightarrow 0} \left[ \sqrt{\alpha(a)/\alpha_c - 1} \ln\left(\frac{r}{a}\right) + \theta(a) \right] = 0. \quad (55)$$

Then we get a running coupling constant same as we obtained earlier by the analysis of the Schwinger-Dyson equation for the electron propagator;

$$\alpha(a) = \alpha_c + \frac{\alpha_c \pi^2}{[\ln(a\mu)]^2}, \quad (56)$$

where  $\mu$  is the renormalization point.

## 2.4 Operator Product Expansion

Let us go back to the differential gap equation, eq. (14). For large momentum,  $p \gg B(p^2)$ , the equation is linearized as

$$p^2 \frac{d^2 B}{dp^2} + 3p \frac{dB}{dp} + rB = 0 \quad (57)$$

where  $r = 3\alpha/\pi$ . When  $r < 1$ , the solutions are for  $p \gg B(p)$

$$B(p) \simeq m_R \left( \frac{\mu}{p} \right)^\epsilon + \frac{\kappa}{p^2} \left( \frac{\mu}{p} \right)^{-\epsilon} \quad (58)$$

The operator product expansion of the propagator in momentum space is

$$\lim_{p^2 \rightarrow \infty} \langle \psi \bar{\psi}(p) \rangle = \frac{A(p/\mu)}{\not{p}} + \frac{C(p/\mu) m_R(\mu) \langle 1 \rangle}{p^2} + \frac{D(p/\mu) \langle \bar{\psi} \psi \rangle}{p^4} + \dots \quad (59)$$

where  $A, C$  and  $D$  are the coefficient functions. If we plug the solutions to OPE,

$$\lim_{p^2 \rightarrow \infty} \langle \psi \bar{\psi}(p) \rangle = \frac{1}{\not{p}} + \frac{B(p)}{p^2} + \dots \quad (60)$$

$$= \frac{1}{\not{p}} + \frac{m_R}{p^2} \left( \frac{\mu}{p} \right)^\epsilon + \frac{\kappa}{p^4} \left( \frac{\mu}{p} \right)^{-\epsilon} + \dots \quad (61)$$

We find that  $m_R$  is the renormalized mass and  $\kappa = \langle \bar{\psi} \psi \rangle$ . As analyzed by Cohen and Georgi, one can show that when  $r < 1$  there is no solution  $\kappa \neq 0$  in the chiral limit  $m_R \rightarrow 0$ . For  $r < 1$ ,  $m_R$  is never zero except for the trivial solution,  $B(p) = 0$ .

When  $r > 1$ , the solutions to the gap equation at large momentum are

$$B(p) = \frac{\kappa}{p} \sin [\sqrt{r-1} \ln(p/\mu) + \varphi] \quad (62)$$

where  $\kappa$  and  $\varphi$  are two parameters that characterize the solution. Now, if we take the non-perturbative running of coupling eq. (26) proposed by Miransky, we get

$$B(p) = \frac{\bar{\kappa}}{p} \quad (63)$$

where  $\bar{\kappa} = \kappa \sin(\pi + \varphi)$ . We see that, for  $r > 1$  with the non-perturbative running coupling, two operators  $m_R$  and  $\bar{\psi}\psi$  coalesce; also, the anomalous dimension of  $\bar{\psi}\psi(x)$  becomes 1. But, as shown earlier, for  $r > 1$  we can take  $m_0 \rightarrow 0$  with keeping  $m = B(0)$  finite and thus  $\bar{\kappa} \neq 0$ . Therefore, one can conclude that the solution found for the quenched QED with non-perturbative running of coupling consistently describes spontaneous breaking of chiral symmetry in the language of operator product expansion.

## 2.5 Electron as a soliton in bosonized QED

In this section we describe the low energy effective action of the strong phase of QED and argue that massive electrons appear as skyrmionic solutions in the effective action [17]. Given that chiral symmetry is spontaneously broken in the strong phase of QED, one may try to construct an low energy effective action for the strong phase. At low energy, the right degrees of freedom are massless Nambu-Goldstone bosons of chiral-symmetry breaking and massless photon. The electrons are massive and decoupled.

We define

$$U_j^i(x) = \lim_{x \rightarrow y} \frac{|x-y|}{\kappa} \bar{q}^i(x) (1 + i\gamma_5) q_j(y), \quad (64)$$

which transforms as

$$U(x) \longrightarrow g_L U(x) g_R^\dagger, \quad (65)$$

where  $g_R$  and  $g_L$  are  $N \times N$  unitary matrices. Here,  $N(> 1)$  is the number of flavors.

In a vacuum  $U$  has a constant (nonzero) expectation value, which we can choose to be  $\langle U_j^i \rangle = \delta_j^i$ . Then, the Nambu-Goldstone bosons are described by

$$U(x) = g_L^\dagger(x)g_R(x), \quad (66)$$

which satisfies  $U^\dagger U = 1$ . Since the current corresponding to the field  $\det U$  is the  $U_A(1)$  current which is anomalous due to the Adler-Bell-Jackiw anomaly, we must impose  $\det U = 1$  and we have  $N^2 - 1$  massless Nambu-Goldstone bosons.

To the lowest order (in derivative expansion) in the effective action is the sum of their kinetic terms:

$$S_{\text{eff}} = \int dx \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} F^2 \text{tr} \partial_\mu U^\dagger \partial_\mu U \right) + \dots \quad (67)$$

Here,  $F$  is the electropion decay constant. The effective action (67) has too much symmetry. It is invariant under two separate discrete symmetries. As Witten[19] has argued in the context of QCD, one has to add the Witten-Wess-Zumino term[20] in the effective action in order for the effective theory has same symmetry as QED:

$$S_{\text{WWZ}} = i \frac{5n}{240\pi^2} \int dx d\tau \epsilon^{\mu\nu\rho\sigma} \text{tr} \tilde{U}^{-1} \frac{\partial \tilde{U}}{\partial \tau} \tilde{U}^{-1} \partial_\mu \tilde{U} \tilde{U}^{-1} \partial_\nu \tilde{U} \tilde{U}^{-1} \partial_\rho \tilde{U} \tilde{U}^{-1} \partial_\sigma \tilde{U}. \quad (68)$$

The coefficient  $n$  of the Witten-Wess-Zumino term has to be integer in order for the effective action to be consistent at quantum level. Furthermore, by calculating the three-point function, whose residue at the  $p^2 = 0$  pole is determined by the Adler-Bell-Jackiw anomaly,

$$\langle J_{\mu 5}(p) J_{\nu 5}(q) J_{\rho 5}(r) \rangle \quad (69)$$

both in the effective action and in QED, we get  $n = 1$ .

Though the Nambu-Goldstone bosons are electrically neutral, it must have coupling with photons, since they are made of charged particles. The coupling term

with lowest number of derivatives is

$$\beta \int dx A_\mu J^\mu, \quad (70)$$

where  $J^\mu$  is the topological current

$$J^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} U^{-1} \partial_\nu U U^{-1} \partial_\rho U U^{-1} \partial_\sigma U. \quad (71)$$

The charge of the topological current corresponds to the winding number of the third homotopy group of  $SU(N)$ . The coefficient  $\beta$  can be determined by comparing the pole at  $p^2 = 0$  in the four-point function

$$\langle \text{tr} [J_{\mu 5}(p) J_{\nu 5}(q) J_{\rho 5}(r) A_\sigma(s)] \rangle \quad (72)$$

in QED and in the effective action. We get  $\beta = e$ , the electric charge.

The final form of our effective action is then

$$S_{\text{eff}} = \int dx \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} F^2 \text{tr} \partial_\mu U^\dagger \partial^\mu U + e A_\mu J^\mu \right] + S_{\text{WWZ}}. \quad (73)$$

We look for a solitonic excitation in the effective action. We take the following Ansatz for the static and finite-energy solution of the winding number 1:

$$U_c(x) = e^{i\vec{\tau} \cdot \hat{x} \theta(r)}, \quad A_0 = \omega(r), \quad A_i = 0, \quad (74)$$

where  $\vec{\tau}$  is the generator of  $SU(2)$  subgroup of  $SU(N)$ .

The energy of this static configuration is

$$E(\omega, \theta) = \int 4\pi r^2 dr \left[ -\frac{1}{2} \omega'^2 + F^2 \left( \theta'^2 + 2 \frac{\sin^2 \theta}{r^2} \right) + \frac{e}{2\pi^2} \frac{\omega}{r^2} \sin^2 \theta \theta' \right]. \quad (75)$$

Using Gauss' theorem,  $\omega' = Q(r)/4\pi r^2$ , one can eliminate  $\omega$  in terms of  $\theta$ . Then, one can get a lower bound on the energy, which is analogous to the Bogomolny bound for monopoles. We find

$$E > \frac{\pi}{\sqrt{2}} e F. \quad (76)$$



The upper bound can be obtained using the variational method. We get

$$E < 3.3eF. \quad (77)$$

Therefore, we find there is a static configuration of finite energy. Now, we try to find its quantum number. To this end, we have to quantize the zero modes of the soliton.

If  $U(x)$  is a solution, so is  $AUA^{-1}$  for any  $A \in SU(N)$ .  $A_1$  and  $A_2$  are equivalent if  $A_1 = A_2h$  and  $h$  is in the commutant  $U(N-2)$  of  $SU(2)$  in  $SU(N)$ . Therefore, we see that  $A$  belong to the coset space  $M = SU(N)/U(N-2)$ . Right multiplication of  $A$  by  $h \in SU(2)$  corresponds to spatial rotations and the left multiplication by  $g \in SU(N)$  to flavor transformations.

The effective Lagrangian for the zero modes  $A$  can be obtained by substituting  $U(\vec{x}, t) = A(t)U_c(\vec{x})A(t)^{-1}$  into (73):

$$L[A] = -E + \frac{1}{2}I_{\alpha\beta}\text{tr}\lambda_\alpha A^{-1}\dot{A}\text{tr}\lambda_\beta A^{-1}\dot{A} - i\frac{1}{2}\text{tr}YA^{-1}\dot{A}, \quad (78)$$

where  $E$  is the energy of the static soliton,  $I_{\alpha\beta}$  is an invariant tensor on  $M$  and  $Y$  is a hypercharge,

$$Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0_{(N-2)\times(N-2)} \end{pmatrix} - \frac{1}{N} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{(N-2)\times(N-2)} \end{pmatrix}. \quad (79)$$

Under the transformation  $A(t) \rightarrow A(t)h(t)$  with  $h \in U(N-2)$

$$L \rightarrow L - i\frac{1}{2}\text{tr}Yh^{-1}\dot{h}. \quad (80)$$

It follows that the wavefunction of the soliton must satisfy

$$\begin{aligned} \Psi(Ah) &= \Psi(A) \quad \text{if } h \in SU(N-2) \\ \Psi(AH) &= e^{i(1-2/N)\theta}\Psi(A) \quad \text{if } h = e^{iY\theta}. \end{aligned} \quad (81)$$

The simplest solutions are

$$\Psi(A) = D_{ab}(A), \quad (82)$$

where  $D_{ab}(A)$ ,  $a = 1, \dots, N$  and  $b = 1, 2$  is the matrix representing  $A$  in the fundamental representation of  $SU(N)$ . These wave functions form a doublet under the right action of  $SU(2)$ . Therefore, the ground state of the soliton is a spin-half particle transforming under the fundamental representation of the flavor group. Further more, the particle must be a fermion, since after  $2\pi$  rotation the wavefunction changes sign. We conclude that the soliton in the effective Lagrangian of QED is the electron. Its physical mass is

$$\frac{\pi}{\sqrt{2}}eF < m < 3.3eF. \quad (83)$$

### 3 Large $N$ QED in three dimensions

Three dimensional gauge theories have been studied intensively recently, since they can describe the high temperature limit of 4-dimensional field theories, as well as their relevance to planar condensed matter system.

Quantum electrodynamics in 2+1 dimensions ( $QED_3$ ) is super-renormalizable and has severe IR divergence, which is softened in  $1/N$  expansion, where  $N$  is the number of flavors. In the IR region, the dimensionful coupling,  $\alpha$ , drops out and  $1/N$  behaves like a dimensionless coupling in  $1/N$  expansion of  $QED_3$ . The dynamical symmetry breaking in  $QED_3$  has been studied in the framework of the Schwinger-Dyson equation by Appelquist *et. al.*[6] They showed for even  $N$  that, when  $N < N_c \simeq 32/\pi^2$ , dynamical fermion mass is generated in the chiral limit and the flavor symmetry  $U(2N)$  breaks down to  $U(N) \times U(N)$ : the dynamical mass is obtained as

$$m \sim \alpha \exp \left[ \frac{-2\pi}{\sqrt{N_c/N - 1}} \right], \quad (84)$$

where  $\alpha \equiv e^2/N$ . This result was supported by a lattice calculation[21]. But, there has been a criticism on this work because using the bare vertex together with  $A(p) = 1$  may lead to a result inconsistent with  $1/N$  expansion[22]. To overcome this shortcoming, there has been an attempt to use a (nonlocal) gauge where  $A(p) = 1$  exactly and use an Ansatz for the vertex function to satisfy the WT identity[23]. They found a finite critical number of flavors,  $N_c = 128/3\pi^2$ .

On the other hand, Pisarski [24] argued, using an effective field theory approach with  $\epsilon (= 4 - d)$  expansion, that for any  $N$  the dynamical fermion mass is generated, namely, the critical number of flavor is  $\infty$ .

In this section, I will discuss the dynamical mass generation in  $QED_3$  and the property of the phase transition near  $N_c$ , since it has some resemblance to four-

dimensional quenched QED. At the end I will review the effect of Chern-Simons term on the generation of dynamical mass and the phase transition.

### 3.1 Schwinger-Dyson equation in $QED_3$

The Lagrangian of three dimensional quantum electrodynamics<sup>3</sup> is given as

$$\mathcal{L} = \sum_{i=1}^N \bar{\psi}_i (i\not{\partial} - e\not{A})\psi - \frac{1}{4}F_{\mu\nu}^2 \quad (85)$$

where  $\psi$  is a four-component spinor. The Lagrangian has  $U(2N)$  symmetry generated by

$$J_a^\mu = \bar{\psi}\gamma^\mu T_a \psi, \quad J_{a5}^\mu = \bar{\psi}\gamma^\mu T_a \gamma^5 \psi \quad (86)$$

$$J_{a3}^\mu = \bar{\psi}\gamma^\mu \gamma^3 T_a \psi, \quad J_{a35}^\mu = \bar{\psi}\gamma^\mu T_a \gamma_3 \gamma^5 \psi \quad (87)$$

To have a well-defined theory for large  $N$  we keep  $\alpha = Ne^2/8$  fixed as  $N \rightarrow \infty$ . The photon propagator in  $1/N$  expansion is obtained by summing up all the bubble diagrams, which is in Landau gauge

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu} - p_\mu p_\nu / p^2}{p^2 [1 + \Pi(p)]} \quad (88)$$

where

$$\Pi(p) = \frac{\alpha}{8p}. \quad (89)$$

The leading order Schwinger-Dyson equation is in euclidean notation

$$A(p) = 1 + \left( \frac{1}{N} \right) \quad (90)$$

$$B(p) = \frac{8\alpha}{N} \int_k \frac{\gamma^\mu D_{\mu\nu}(p-k) B(k) \gamma^\nu}{k^2 + B^2(k)} \quad (91)$$

Angular integration in eq. (91) gives

$$B(p) = \frac{4\alpha}{\pi^2 N p} \int dk \frac{k B(k)}{k^2 + B^2(k)} \ln \left[ \frac{k+p+\alpha}{|k-p|+\alpha} \right] \quad (92)$$

---

<sup>3</sup> Here we discuss a parity-invariant theory

using the formular

$$\int_0^\pi \sin \theta d\theta \frac{1}{(p^2 + k^2 - 2pk \cos \theta) \left(1 + \frac{\alpha}{\sqrt{p^2 + k^2 - 2pk \cos \theta}}\right)} = \frac{1}{pk} \ln \left[ \frac{p + k + \alpha}{|p - k| + \alpha} \right] \quad (93)$$

Since  $QED_3$  is super-renormalizable, the integral eq. (93) is rapidly damped for  $p > \alpha$ . For  $p < \alpha$ , it takes the approximate form

$$B(p) = \frac{4}{\pi^2 N p} \int_0^\alpha dk \frac{k B(k)}{k^2 + B^2} (k + p - |k - p|) \quad (94)$$

where  $\alpha$  is used as a UV cut-off and the leading term in the logarithm is kept:

$$\ln \left[ \frac{p + k + \alpha}{|p - k| + \alpha} \right] = \left[ \frac{p + k}{\alpha} - \frac{|p - k|}{\alpha} \right] \quad (95)$$

Eq. (94) can be replaced by the differntial equation

$$\frac{d}{dp} \left[ p^2 \frac{dB(p)}{dp} \right] = - \left[ \frac{8}{\pi^2 N} \right] \frac{p^2 B(p)}{p^2 + B^2(p)} \quad (96)$$

together with the boundary conditions

$$0 \leq B(0) < \infty \quad (97)$$

and

$$\lim_{p \rightarrow \alpha} \left[ p \frac{dB(p)}{dp} + B(p) \right] = 0 \quad (98)$$

The boundary condition eq.(98) insures the absence of a bare mass. In a regime  $p \gg B(p)$ , as in the case of quenched  $QED_4$ , the eq. (96) is linearized by replacing  $B(p)$  with  $B(0)$  in the denominator to give a power solution

$$B(p) \sim p^a \quad (99)$$

where

$$a = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{32}{\pi^2 N}} \quad (100)$$

For  $N > 32/\pi^2$ , the power solution does not satisfy the boundary condition eq. (98).

But, for  $N < 32/\pi^2$ , it does and the fermion self energy is oscillatory,

$$B(p) = \frac{\kappa}{\sqrt{p}} \sin \left[ \frac{1}{2} \sqrt{32/\pi^2 N - 1} \ln \left( \frac{p}{B(0)} \right) + \varphi \right], \quad (101)$$

which satisfies eq. (98) if

$$\frac{1}{2} \sqrt{32/\pi^2 N - 1} \ln \left( \frac{\alpha}{B(0)} \right) + \varphi = n\pi \quad (102)$$

Thus we find

$$B(0) \simeq \alpha \exp \left[ \frac{-2(n\pi - \varphi)}{\sqrt{32/\pi^2 N - 1}} \right] \quad (103)$$

We see that for  $N < N_c = 32/\pi^2$  the fermions get dynamical mass which vanishes smoothly as  $N \rightarrow N_c$ . It looks like the phase transition is of second-order.

### 3.2 Quantum Phase Transition in $QED_3$

Pisarski studied an effective field theory which has same symmetry as  $QED_3$  with  $N$  massless four-component spinors:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \text{tr} (\partial_\mu \phi)^2 + \frac{8\pi^2 \mu^\epsilon}{4!} \left\{ g_1 [\text{tr}(\phi^2)]^2 + g_2 \text{tr}(\phi^4) \right\} \quad (104)$$

where all renormalizable terms are kept but all cubic couplings and quadratic couplings are absent in the effective Lagrangian, assuming the multicritical point.

$\phi$  is a traceless  $N \times N$  Hermitian matrix and transforms under  $SU(N)$  as

$$\phi \rightarrow U^\dagger \phi U. \quad (105)$$

Note that the trace degree of freedom, which is parity-odd, is massive and decoupled at the multicritical point since it can not have v.e.v due to Vafa-Witten theorem [25].

He showed that, for  $N > \sqrt{5}/2$ , there are no IR stable fixed points by analyzing the  $\beta$ -functions for  $g_1$  and  $g_2$  obtained by  $\epsilon$ -expansion. He then concluded that mass must be generated and thus fermions are massive for all  $N$ . Therefore, if the effective theory described by eq.(104) is in a universality class with  $QED_3$ , dynamical fermion mass is generated for all  $N$  in  $QED_3$ .

To examine the basic assumptions of Pisarski argument, Appelquist, Terning and Wijewardhana [26] studied the fermion-antifermion bound state in the symmetric phase of  $QED_3$ . They solved the Schwinger-Dyson equations for fermion-antifermion scattering amplitude in  $1/N$  expansion. The Schwinger-Dyson equation is

$$T_{\lambda\rho\sigma\tau}(p, q) = \frac{16\alpha}{N} (\gamma^\mu)_{\sigma\lambda} D_{\mu\nu}(p) (\gamma^\nu)_{\rho\tau} \quad (106)$$

$$+ \int_k T_{\lambda\rho\sigma'\tau'}(k, q) \frac{1}{\frac{1}{2}q + \not{k}} (\gamma^\mu)_{\sigma\sigma'} D_{\mu\nu}(p - k) (\gamma^\nu)_{\tau'\tau} \frac{1}{\frac{1}{2}q + \not{k}} \quad (107)$$

The scattering amplitude can be written as

$$T_{\lambda\rho\sigma\tau}(p, q) = \delta_{\lambda\rho} \delta_{\sigma\tau} T(p, q) + \dots \quad (108)$$

where the ellipsis indicates pseudoscalar, vector, axial vector, and tensor components. The scalar channel factorizes in the SD equations and gives

$$T(p, q) = \frac{16\alpha}{3Np(p + \alpha)} + \frac{16\alpha}{3\pi^2 Np} \int \frac{dk}{k} T(k, q) \ln \left[ \frac{k + p + \alpha}{|k - p| + \alpha} \right]. \quad (109)$$

Since we are interested in the massless bound-state at the critical point we try to see if there exists a pole as  $q \rightarrow 0$ . Since  $QED_3$  damps rapidly as  $p \gg \alpha$  and  $q$  acts as an infrared cutoff, we use the approximation:

$$T(p, q) = \frac{16}{3Np} + \frac{16}{3\pi^2 Np} \int_q^\alpha \frac{dk T(k, q)}{k} (k + p - |k - p|). \quad (110)$$

For  $p > q$ , eq. (110) can be converted to a differential equation:

$$p \frac{d^2}{dp^2} (pT) = -\frac{32}{3\pi^2 N} T \quad (111)$$

The solutions of eq. (111) are

$$T(p, q) = \frac{A(q)}{\alpha} \left(\frac{p}{\alpha}\right)^{-\frac{1}{2}+\frac{1}{2}\eta} + \frac{B(q)}{\alpha} \left(\frac{p}{\alpha}\right)^{-\frac{1}{2}-\frac{1}{2}\eta} \quad (112)$$

where

$$\eta = \sqrt{1 - N_c/N} \quad (113)$$

and  $N_c = 128/3\pi^2$ . The coefficient  $A$  and  $B$  are determined as

$$A = \frac{-\left(\frac{1}{2} - \frac{1}{2}\eta\right)^2 \pi^2 \left(\frac{q}{\alpha}\right)^{-\frac{1}{2}+\frac{1}{2}\eta}}{2\left(\frac{1}{2} + \frac{1}{2}\eta\right) \left(1 - \left(\frac{1-\eta}{1+\eta}\right)^2 \left(\frac{q}{\alpha}\right)^\eta\right)}, \quad (114)$$

$$B = \frac{\left(\frac{1}{2} - \frac{1}{2}\eta\right) \pi^2 \left(\frac{q}{\alpha}\right)^{-\frac{1}{2}+\frac{1}{2}\eta}}{2\left(1 - \left(\frac{1-\eta}{1+\eta}\right)^2 \left(\frac{q}{\alpha}\right)^\eta\right)}. \quad (115)$$

The amplitude has a pole at  $q_0$  in the complex  $q$  plane with

$$|q_0| = \alpha \left(\frac{1+\eta}{1-\eta}\right)^{\frac{2}{\eta}} \quad (116)$$

In the limit  $\eta \rightarrow 0$ ,

$$|q_0| \rightarrow \alpha e^4 \quad (117)$$

We see that there is no pole at the critical point. Thus the assumption of the existence of muticritical point in Pisarki's analysis is not applicable in the case of large  $N$   $QED_3$ . The phase transition near  $N_c$  seems quite peculiar: in the broken phase the dynamical fermion mass, being an order parameter for the phase transition, vanishes smoothly as we approach the critical point, while in the symmetric phase the mass of the bound state does not vanish near the critical point.

### 3.3 $QED_3$ with a Chern-Simons term

In 2+1 dimensional gauge theories the gauge bosons can have parity-violating mass by Chern-Simons term for the gauge fields. Since the Chern-Simons term breaks



parity, it induces not only parity-violating mass to fermion but also fractional spin. Here I will briefly discuss the effect of the Chern-Simons term in dynamical generation of parity-even mass and also in the phase transition of  $QED_3$ . As studied by Hong and Park [27], if Chern-Simons term is added in  $QED_3$ , it tends to break parity maximally. Namely, the dynamical parity-even fermion mass gets reduced in the magnitude and the critical coupling ( $1/N_c$ ) gets larger. Later, Kondo and Maris [28] analyzed this model in a nonlocal gauge for which  $A(p) = 1$  and found similar results. Analyzing the model for small coefficient of Chern-Simons term, They showed the dynamical parity-even fermion mass does not vanish as  $N_c \rightarrow N$ , indicating that the phase transition is first order in contrast to the phase transition in pure  $QED_3$ .

One can study the fermion-antifermion scattering amplitude in the symmetric phase of  $QED_3$  in the presence of Chern-Simons term, following the analysis of Appelquist *et. al* for pure  $QED_3$  [29]. Using the coefficient of the Chern-Simons term as an expansion parameter, it is found that the mass of the scalar channel bound state vanishes as we approach the critical point, which shows that in the presence of Chern-Simons term  $QED_3$  exhibits a peculiar phase transition as in the case of pure  $QED_3$ .

## 4 Conclusion

As we have seen in the analysis of Abelian gauge theory in three and four dimensions, chiral symmetry is dynamically broken when the coupling is larger than a critical coupling.

In four dimensions, the critical coupling is  $\pi/3$  in Landau gauge and it turns out to be the UV fixed point. The  $\beta$ -function is due to the wave-function collapse of the electron-positron bound state. Quenched  $QED_4$  exhibits two-phase struc-

ture. In the strong phase, chiral symmetry is spontaneously broken and the low energy spectrum contains the massless Nambu-Goldstone bosons, massless photon, and massive electrons as we have seen in the low energy effective field theory of  $QED_4$ . The massive electron is nothing but a Skymion in the bosonized quantum electrodynamics.

In three dimensions,  $1/N$  expansion is a sensible way to study nonperturbative phenomena in  $QED_3$ , since it is super renormalizable and has severe IR divergence in ordinary coupling perturbation. If one uses  $1/N$  expansion,  $1/N$  becomes an effective coupling of electron and positron in IR region and dynamical electron mass generates when  $1/N$  is larger than a critical value (or  $N < N_c$ ). The quantum phase structure of  $QED_3$  seems peculiar; The order parameter approaches zero at the critical point in the asymmetric phase but the correlation length remains finite at the critical point in the symmetric phase.

With a Chern-Simons term, the parity of  $QED_3$  tends to break maximally. Namely, the magnitude of parity-even electron mass and the critical number of flavors get reduced. In the asymmetric phase the order parameter remains finite at the critical point while the correlation length gets infinite in the symmetric phase.

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